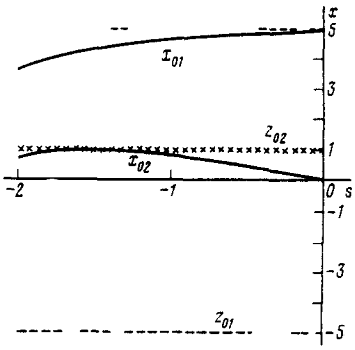


Fig.2

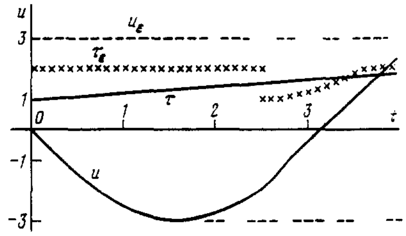


Fig.3

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# THE METHOD OF PURSUIT BY SEVERAL CONTROLLED OBJECTS OF DIFFERENT TYPES*


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The problem of the pursuit of one evader by several controlled objects of different types is examined. The sufficient conditions are obtained for the pursuit game to terminate in a finite time. The proposed method of pursuer interaction assumes that the pursuing players are separated into two groups, the first of which holds the evader in some domain, while the second searches for the evader in this domain. The paper touches on the researches in /1—9/. Typical examples illustrate the results.


Let the motions of the vectors $z_1, \ldots, z_m$ in the $n$-dimensional Euclidean space $R^n$ be described by the equations

$$z_i^{\cdot} = C_i z_i + u_i - v, \quad z_i(0) \approx z_i^{\circ}, \quad i = 1, \ldots, m \tag{1}$$


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where $C_i$ is a constant matrix of dimensions $n \times n$, $u_i \in P_i$, $v \in Q$, $P_i \subset R^{p_i}$, $Q \subset R^q$ are convex compacta. In $R^n$ there are specified the terminal sets $M_1, M_2, \ldots, M_m$, where $M_i = M_i^1 + M_i^2$, $M_i^1$ is a linear subspace in $R^n$, $M_i^2$ is a convex compactum in $L_i^1$, $L_i^1$ is the orthogonal complement to the subspace $M_i^1$ in $R^n$. The data listed above describe a many-person differential game (1) in which a group of pursuers participate having at its disposal a control vector $u = (u_1, u_2, \ldots, u_m)$ and a pursued player at whose disposal is a vector $v$. We will consider a pursuit problem for the differential game (1).

As the evader's strategy we fix a class of programmed measurable functions $v(t) \in Q$, $t \geqslant 0$.

As the $i$-th pursuer's strategy we fix a class of functions

$$U^i(t, v, z^\circ): [0, \infty) \times Q \times R^{mn} \to R^{p_i}$$

for which the following conditions are satisfied:

1) $U^i(t, v, z^\circ) \in P_i$ for all $t \geqslant 0$, $v \in Q$;

2) the function $u_i(t) = U^i(t, v(t), z^\circ)$ is Lebesgue-measurable in the interval $[0, \infty)$ for an arbitrary measurable $v(t) \in Q$, $t \geqslant 0$.

Following /5/, the strategies $U^i(t, v, z^\circ)$ are called stroboscopic. The pursuit strategy is the vector $u(t) = (u_1(t), \ldots, u_m(t))$, where $u_i(t)$, $i = 1, \ldots, k$ is the $i$-th pursuer's stroboscopic strategy, $u_i(t)$, $i = k+1, \ldots, m$ is the $i$-th pursuer's programmed strategy, $0 \leqslant k \leqslant m$ (if $k = 0$, $u_i(t)$ $(i = 1, \ldots, m)$ are the $i$-th pursuer's programmed strategies).

The pursuit problem can be formulated as follows. It is required to find a condition for the parameters of game (1), for which there exists, for prescribed initial states $z_i^\circ$ $(i = 1, \ldots, m)$ a pursuit strategy $u^*(t) = (u_1^*(t), \ldots, u_m^*(t))$ such that at least one vector $z_i(t)$ $(i = 1, \ldots, m)$, which is a solution of the equation

$$z_i^{\cdot} = C_i z_i + u_i^*(t) - v(t), \quad z_i(0) = z_i^\circ$$

arrives at the corresponding terminal set $M_i$ no later than some finite instant of time.

Let $\pi_i$ be the orthogonal projection operator from $R^n$ onto $L_i^1$, and let $\overset{*}{-}$ be the geometric difference of the sets /1/.

*Condition A* /1, 7/. Upper-semicontinuous functions

$$\varkappa_i(t, \tau): [0, \infty) \times [0, t] \to R^1, \quad \varkappa_i(t, \tau) \geqslant 0, \int_0^t \varkappa_i(t, \tau)\, d\tau = 1$$

and sets $M_i^3$, $M_i^2 \overset{*}{-} M_i^3 \neq \varnothing$, exists such that the sets

$$w_i(t, \tau) = (-\varkappa_i(t, \tau) M_i^3 + \pi_i \exp(t-\tau) C_i \cdot P_i) \overset{*}{-} \pi_i \exp(t-\tau) C_i \cdot Q$$

are non-empty for all $\tau, t, \tau \in [0, t]$, $t \geqslant 0$

From Condition $A$ and the conditions for game (1) it follows that a Borel-measurable function $\beta_i(t, \tau): [0, \infty) \times [0, t] \to R^n$, $\beta_i(t, \tau) \in w_i(t, \tau)$ exists. Let $k$ be an integer, $0 \leqslant k \leqslant m$, defined by the following condition: the parameters defining game (1) satisfy Condition $A$ for $i = 1, \ldots, k$; they do not satisfy Condition $A$ for $i = k+1, \ldots, m$. If Condition $A$ cannot be satisfied for any $i$, we take $k = 0$.

Let $M_i^4 \subset M_i^2 \overset{*}{-} M_i^3$ and $m_i^4 \in M_i^4$. For $i = 1, \ldots, k$ we consider the functions

$$\varphi_i(t, z_i^\circ, m_i^4) = \pi_i \exp tC_i \cdot z_i^\circ - m_i^4 + \int_0^t \beta_i(t, \tau)\, d\tau$$

If a number $i$ and a positive constant $T$ exist such that $\varphi_i(T, z_i^\circ, m_i^4) = 0$, then the pursuit problem is solvable at the instant $T$ for game (1) in position $z^\circ$ /1/. Therefore, without loss of generality, we will henceforth assume that $\varphi_i(t, z_i^\circ, m_i^4) \neq 0$ for all $i, t, m_i^4, t \geqslant 0, m_i^4 \in M_i^4$. We set /7, 8/

$$\begin{gathered}\lambda(i, t, \tau, v, z_i^\circ, m_i^4) = \max\{\lambda: \lambda \geqslant 0, \; -\lambda\varphi_i(t, z_i^\circ, m_i^4) \in \\ (-\varkappa_i(t, \tau) M_i^3 + \pi_i \exp(t-\tau) C_i \cdot P_i - \pi_i \exp(t-\tau) C_i \cdot v - \\ \beta_i(t, \tau))\}, \quad \lambda(i, t, \tau, v, z_i^\circ, M_i^4) = \max_{m_i^4 \in M_i^4} \lambda(i, t, \tau, v, z_i^\circ, m_i^4)\end{gathered} \tag{2}$$

If $k = m$, the sufficient conditions for the pursuit problem to be solvable in such a game have been formulated, for example, in /9/. We shall assume $k < m$. Let $T$ be an arbitrary positive constant, and $u_j(t)$ be arbitrary programmed controls, $j = k+1, \ldots, m$, $0 \leqslant t \leqslant T$. We put

$$E(t, T, z^\circ) = \Big\{ v(\tau): \; 0 \leqslant \tau \leqslant t, \; v(\tau) \in Q, \int_0^t \lambda(i, T, \tau, v(\tau), z_i^\circ, M_i^4)\, d\tau < 1, \quad i = 1, \ldots, k \Big\}$$

$$q_j(t, v(\cdot)) = -\int_0^t \pi_j \exp(t-\tau) C_i \cdot v(\tau)\, d\tau$$

$$V_j(t, T, z^\circ) = \{q_j(t, v(\cdot)) \colon v(\tau) \in E(t, T, z^\circ)\}$$

$$N_j(t) = -\pi_j \exp tC_j \cdot z_j^\circ - \int_0^t \pi_j \exp(t-\tau) C_j \cdot u_j(\tau)\, d\tau + M_j^2$$

$$N(t) = \bigcup_{j=k+1}^{m} N_j(t), \quad t \leqslant T, \quad j = k+1, \ldots, m$$

*Assumption 1.* The vectors $q_j(t, v(\cdot))$ are independent of $j$ for all $v(\tau) \in E(t, T, z^0)$, $0 \leqslant \tau \leqslant t \leqslant T$; $L_j^1 = R^\nu$.

We set

$$V(t, T, z^\circ) = V_j(t, T, z^\circ), \quad q(t, v(\cdot)) = q_j(t, v(\cdot))$$

*Assumption 2.* For the position $z^\circ = (z_1^\circ, \ldots, z_m^\circ)$ a positive constant $T$ and admissible controls $u_j(t)$, $0 \leqslant t \leqslant T$, $j = k+1, \ldots, m$ exist, such that for some instant $t^* \leqslant T$, $N(t^*) \supset V(t^*, T, z^\circ)$.

*Condition B.* We saw that Condition $B$ is satisfied in the interval $[T_1, T_2]$, $0 \leqslant T_1 \leqslant T_2 \leqslant T$ for the sets $N(t)$, $V(t, T, z^\circ)$ (the condition for the set $N(t)$ to work its way through the set $V(t, T, z^\circ)$ in the interval $[T_1, T_2]$), if a continuous function $\xi(x, t) \colon R^\nu \times [T_1, T_2] \to R^1$ exists such that

a) $V(T_1, T, z^\circ) \in \{x \colon \xi(x, T_1) \leqslant 0\}$; $V(T_1 T, z^\circ) \cap \{x \colon \xi(x, T_1) = 0\} \neq \varnothing$;

b) $N(t) \supset V(t, T, z^\circ) \cap \{x \colon \xi(x, t) = 0\}$ for all $t \in [T_1, T_2]$;

c) $N(T_2) \supset V(T_2, T, z^\circ) \cap \{x \colon \xi(x, T_2) \leqslant 0\}$.

*Assumption 3.* For a position $z^\circ$ controls $u_j(t)$, $j = k+1, \ldots, m$ and positive constants $T_1$, $T_2$, $T$, $T_1 \leqslant T_2 \leqslant T$ exist, such that Condition $B$ is satisfied in the interval $[T_1, T_2]$ for the sets $N(t)$, $V(t, T, z^\circ)$.

*Theorem.* If Assumptions 1, 2 or 1, 3 are satisfied at position $z^\circ$, then the pursuit problem is solvable up to the instant $T$ for the position $z^\circ$.

*Proof.* Suppose Assumptions 1 and 2 are satisfied for position $z^\circ$. We consider the function $\lambda(i, t, \tau, v, z_i^\circ, m_i^4)$ defined by relation (2). According to (2), the function $\lambda(i, t, \tau, v, z_i^\circ, m_i^4)$ is upper-semicontinuous in the collection of arguments $v$, $m_i^4$ for fixed $\tau$ and is Borel-measurable in $\tau$ for fixed $v$, $m_i^4$ (the arguments $i$, $t$, $z_i^\circ$ are fixed). Consequently /10/, the compact set

$$\Lambda_i^4(t, \tau, v) = \{m_i^4 \colon m_i^4 \in M_i^4, \ \lambda(i, t, \tau, v, z_i^\circ, m_i^4) = \lambda(i, t, \tau, v, z_i^\circ, M_i^4)\}$$

is upper-semicontinuous by inclusion in $v$ for fixed $\tau$ and is Borel-measurable in $\tau$ for fixed $v$, and a Borel-measurable function $m_i^4(t, \tau, v) \in \Lambda_i^4(t, \tau, v)$ exists. Let $v(t)$ be an arbitrary programmed control of the evader. We direct the $i$-th pursuer ($i = 1, \ldots, k$) to construct his own control $u_i(t)$ at the instant $t$, $t \in [0, T]$, in the following manner. If at the instant $t \geqslant 0$

$$\rho_i(t; v(\tau), 0 \leqslant \tau \leqslant t) = 1 - \int_0^t \lambda(i, T, \tau, v(\tau), z_i^\circ, M_i^4)\, d\tau > 0$$

then the functions $m_i^3(t) \in M_i^3$, $u_i(t) \in P_i$ are a solution of the equation

$$-\varkappa_i(T, t) m_i^3(t) + \pi_i \exp(T-t) C_i \cdot (u_i(t) - v(t)) - \beta_i(T, t) = -\lambda(i, T, t, v(t), z_i^\circ, M_i^4) \times \left(\pi_i \exp TC_i \cdot z_i^\circ - m_i^4(T, t, v(t)) + \int_0^T \beta_i(T, \tau)\, d\tau\right) \tag{3}$$

If $t_1^i$ is the first instant when $\rho_i(t_1^i, v(\tau), 0 \leqslant \tau \leqslant t_1^i) = 0$, then for $t \in (t_1^i, T]$ the functions $m_i^3(t) \in M_i^3$, $u_i(t) \in P_i$ are a solution of the equation

$$-\varkappa_i(T, t) m_i^3(t) + \pi_i \exp(T-t) C_i \cdot (u_i(t) - v(t)) - \beta_i(T, t) = 0 \tag{4}$$

By virtue of (1)–(2) and Condition $A$ one of many solutions of (3) and (4) exist. By construction, $\lambda(i, T, t, v(t), z_i^\circ, M_i^4)$, $m_i^4(T, t, v(t))$ are measurable functions of $t$ for fixed $i$, $T$, $z^\circ$.

Consequently, by virtue of Filippov's theorem /10/, (3) and (4) are solvable in the class of measurable functions. We direct the $j$-th pursuer ($j = k+1, \ldots, m$) to choose the controls $u_{k+1}(t), \ldots, u_m(t)$ for which Assumption 2 is satisfied.

By applying strategies $u_i(t)$ the pursuers can guarantee the termination of pursuit by instant $T$. If $q(t, v(\cdot)) \overline{\in} V(t, T, z^\circ)$ for at least one $t \leqslant t^*$, then the evader at instant $T$ is caught by one of the pursuers numbered $i = 1, \ldots, k$. Indeed, in this case we can find a number $i$ such that

$$\int_0^t \lambda(i, T, \tau, v(\tau), z_i^\circ, M_i^4)\, d\tau \geqslant 1$$

We will denote by $t_*$ the first instant t when

$$\int_0^{t_*} \lambda(i, T, \tau, v(\tau), z_i^\circ, M_i^4)\, d\tau = 1$$

Using (3), (4) and the Cauchy formula for (1), we obtain

$$\pi_i z_i(T) = \Big(\pi_i \exp TC_i \cdot z_i^\circ + \int_0^T \beta_i(T, \tau)\, d\tau\Big)\Big(1 - \int_0^{t_*} \lambda(i, T, \tau, v(\tau), z_i^\circ, M_i^4)\, d\tau\Big) + \int_0^{t_*} \lambda(i, T, \tau, v(\tau), z_i^\circ, M_i^4)\, m_i^4(T, \tau, v(\tau))\, d\tau + \int_0^T \varkappa_i(T, \tau)\, m_i^3(\tau)\, d\tau \in M_i^2 \tag{5}$$

i.e., the $i$-th pursuer catches the evader at instant $T$.

If $q(t, v(\cdot)) \in V(t, T, z^\circ)$ for all $t \leqslant t^*$, then at the instant $t^* \leqslant T$ the evader is caught by one of the pursuers numbered $j = k+1, \ldots, m$ by virtue of Assumption 2.

Suppose that Assumptions 1 and 3 are satisfied for position $z^\circ$. We direct the $i$-th pursuer ($i = 1, \ldots, k$) to construct his own control $u_i(t)$ as a solution of Eqs. (3), (4), and the $j$-th pursuer ($j = k+1, \ldots, m$) to choose the controls $u_{j+1}(t), \ldots, u_m(t)$, for which Assumption 3 is satisfied. We will show that by applying strategies $u_i(t)$ the pursuers can guarantee that the pursuit terminates by instant $T$. Let $v(t)$ be an arbitrary programmed control of the evader. If $q(t_*, v(\cdot)) \overline{\in} V(t_*, T, z^0)$ for at least one $t_* \leqslant T$, then the evader at the instant $T$ is caught by one of the pursuers numbered $i = 1, \ldots, k$ (see (5)). For all instants $t \leqslant T$ let $q(t, v(\cdot)) \in V(t, T, z^\circ)$. For such a control $v(t)$, $t \in [0, T]$, two cases are possible: either for all $t$: $t \in [T_1, T_2]$

$$q(t, v(\cdot)) \in V(t, T, z^\circ) \cap \{x: \xi(x, t) \leqslant 0\}$$

or an instant $t^*$: $t^* \in [T_1, T_2]$ exists such that

$$q(t^*, v(\cdot)) \in V(t^*, T, z^\circ) \cap \{x: \xi(x, t^*) > 0\}$$

In the first case, according to Assumption 3 and paragraphs b) and c) of Condition $B$, the evader is caught by one of the pursuers numbered $k+1, \ldots, m$ at instant $T_2$. In the second case we have

$$\xi(q(T_1, v(\cdot)), T_1) \leqslant 0, \quad \xi(q(t^*, v(\cdot)), t^*) > 0$$

whence by virtue of the continuity of $\xi$ in all arguments it follows that an instant $\theta \in [T_1, t^*]$ exists such that $\xi(q(\theta, v(\cdot)), \theta) = 0$. On the strength of Assumption 3 and of item b) of Condition $B$, this signifies that the evader is caught by one of the pursuers numbered $k+1, \ldots, m$ at instant $\theta$. The theorem is proved.

*Notes.* $1^\circ$. Assumption 1 is satisfied for example, for a class of games such as the following. The equations of motion of the pursuers are $x_i^\cdot = A_i x_i + u_i$, $u_i \in P_i$, $i = 1, \ldots, m$, $x_i, u_i \in R^n$, the equations of motion of the evader are $y^\cdot = By + v$, $v \in Q$, $y, v \in R^n$. $P_i, Q$ are convex compacta, and $A_i, B$ are matrices of appropriate sizes. The game terminates if for at least one $i$ we have, at some instant $t$, $\pi_i x_i \cdot (t) + M_i \supset \pi_i y(t)$, where $\pi_i$ is an $(l \times n)$-matrix, $1 \leqslant l \leqslant n$, $M_i$ is a convex compactum in $R^l$. In this case, $\pi_i = \pi$, $i = k+1, \ldots m$ corresponds to Assumption 1.

$2^\circ$. Condition $B$ (the condition for set $N(t)$ to work its way through set $V(t, T, z^\circ)$ in the interval $[T_1, T_2]$) is the condition for the following controllability problem to be solvable. At each instant $t \in [T_1, T_2]$ the points

$$-\pi_j \exp tC_i \cdot z_i^\circ - \int_0^t \pi_j \exp(t-\tau)\, C_j \cdot u_j(\tau)\, d\tau = \alpha_j(t)$$

must be located on a continuous surface $\{x: \xi(x, t) = 0\}$ such that the covering problem

$$\bigcup_{j=k+1}^{m} (\alpha_j(t) + M_j^2) \supset V(t, T, z^\circ) \cap \{x: \xi(x, t) = 0\}$$

is solvable and the conditions

$$N(T_2) \supset V(T_2, T, z^\circ) \cap \{x: \xi(x, T_2) \leqslant 0\}$$

are satisfied at the instant $T_2$. The surface $\{x: \xi(x, t) = 0\}$, $t \in [T_1, T_2]$ must satisfy two conditions: for the set $V(t, T, z^\circ)$, for any continuous curve $\psi(t) \in V(t', T, z^\circ)$, $t \in [T_1, T_2]$, such that $\xi(\psi(t_0), t_0) > 0$, $\xi(\psi(t_1), t_1) > 0$, $T_1 \leqslant t_0 \leqslant t_1 \leqslant T_2$, an instant $t_* \in [t_0, t_1]$ exists such that $\xi(\psi(t_*), t_*) = 0$; b) its intersection with $V(t, T, z^\circ)$ can be covered by sets $N_j(t)$ under certain admissible controls $u_j(t)$.

To solve the covering problem we can use the results on lattice coverings by spheres, generalized cylinders or other convex bodies /11, 12/. A solution of the covering problem is information on the set $\Omega_j(t)$ of all possible locations of the lattice nodes $\alpha_j(t) \in \Omega_j(t)$ for which the problem of covering the set $V(t, T, z^\circ) \cap \{x: \xi(x, t) = 0\}$ by the sets $N_j(t)$, $j = k+1, \ldots, m$ is solvable. The controllability problem then consists of answering the question of whether it is possible to retain $\alpha_j(t)$ in the set $\Omega_j(t)$ for certain admissible $u_j(t)$ in the interval $[T_1, T_2]$.

We present some examples of choosing the surface $\{x: \xi(x, t) = 0\}$ and the controls $u_j(t)$ which ensure that Assumptions 1—3 are satisfied.

*Example 1.* In (1) let

$$C_i = 0,\ n = 2,\ P_i = S_{\rho_i}{}^2(0),\ Q = S_\sigma{}^2(0),\ M_i{}^1 = \{0\},\ M_i{}^2 = S_{l_i}{}^2(0)$$
$$\pi_i = E,\ i = 1, \ldots, m;\ \rho_i = \sigma,\ l_i = 0,\ i = 1, \ldots, k;\ \rho_i < \sigma,\ l_j > 0$$
$$j = k+1, \ldots, m$$

($E$ is a unit $2 \times 2$ matrix). Assumption 1 is satisfied in the case being examined; by (2) we have

$$\lambda(i, t, \tau, v, z_i^\circ, 0) = [(z_i^\circ, v) + ((z_i^\circ, v)^2 + \|z_i^\circ\|^2 (\sigma^2 - (v, v)))]^{1/2} / \|z^\circ_i\|^2,$$
$$i = 1, \ldots, k$$

$$V(t, T, z^\circ) \in K(t, T, z^\circ) = \left\{-\int_0^t v(\tau)\, d\tau : \left(z_i^\circ, -\int_0^t v(\tau)\, d\tau\right) > -\frac{\|z_i^\circ\|^2}{2},\quad i = 1, \ldots, k\right\}$$

Thus, in the case in hand the set $V(t, T, z^\circ)$ belongs to the convex polyhedral set

$$K = \left\{z: \left(z, \frac{z_i^\circ}{\|z_i^\circ\|}\right) > -\frac{\|z_i^\circ\|}{2},\quad i = 1, \ldots, k\right\}$$

We present three typical cases where $N(t)$ works its way through $V(t, T, z^\circ)$ (inside $K = K(t, T, z^\circ)$).

A) $k = 3$, $m = 5$; $\rho_i = \sigma = 1$, $i = 1, 2, 3$; $\rho_j = 1/2$, $l_j = 1\,{}^3/_5$, $j = 4, 5$; $z_1^\circ = (2,0)$, $z_2^\circ = (3,2)$, $z_3^\circ = (-2,1)$, $z_4^\circ = (3, -7)$, $z_5^\circ = (-3\,{}^1/_2, -10\,{}^1/_2)$. The set $K = \{z: z_1 > -1,\ 3z_1 + 2z_2 > -6{}^1/_2,\ -2z_1 + z_2 > -2{}^1/_2\}$. We put

$$\xi(z, t) = z_2 + {}^1/_2 t - 10{}^1/_2,\quad t \geqslant 7,\ T_1 = 7.$$

The controls $u_4(t)$, $u_5(t)$, $0 \leqslant t \leqslant 7$, which for $t = T_1 = 7$ ensure that item a) of Condition $B$ is satisfied have the form $u_4(t) = (-{}^1/_2, 0)$, $u_5(t) = (0, {}^1/_2)$, while the controls $u_4(t)$, $u_5(t)$ which ensure that items b) and c) of Condition $B$ are satisfied have the form $u_4(t) = (0, {}^1/_2)$, $u_5(t) = (0, {}^1/_2)$, $7 \leqslant t \leqslant 27$, $T_2 = 27$.

We note that in this position the pursuit game is solvable without the participation of pursuer number 2; however, his presence enables the pursuers to reduce the capture time.

B) $k = 2$, $m = 4$; $\rho_i = \sigma = 1$, $i = 1, 2$; $\rho_j = {}^1/_2$, $l_j = {}^3/_4$, $j = 3, 4$; $z_1^\circ = (0, -1)$, $z_2^\circ = (0, 2)$, $z_3^\circ = (-5, 2\,{}^3/_4)$, $z_4^\circ = (5, -2\,{}^1/_4)$. The set $K = \{z: -1 < z_2 < {}^1/_2\}$. We put

$$\xi(z, t) = z_1^2 + (z_2 + {}^1/_4)^2 - (7\,{}^1/_2 - {}^1/_2\, t)^2,\quad t \geqslant 5,\quad T_1 = 5$$

The controls $u_3(t)$, $u_4(t)$, $0 \leqslant t \leqslant 5$, which for $t = T_1 = 5$ ensure that item a) of Condition B is satisfied have the form $u_3(t) = (0, -{}^1/_2)$, $u_4(t) = (0, {}^1/_2)$, while the controls $u_3(t)$, $u_4(t)$ which ensure that items b) and c) of Condition B are satisfied have the form

$$u_3(t) = ({}^1/_2, 0),\ u_4(t) = (-{}^1/_2, 0),\ 5 \leqslant t \leqslant 15,\ T_2 = 15$$

C) $k = 0$, $m = 8$, $\sigma = 1$, $\rho_j = {}^1/_2$, $l_j = 4$, $j = 1, \ldots, 8$;

$$z_1^\circ = (-15, 0),\quad z_2^\circ = \left(-\left(\frac{10}{\sqrt{2}} + 5\right), -\frac{10}{\sqrt{2}}\right),\quad z_3^\circ = (0, -15)$$

$$z_4^\circ = \left(\frac{10}{\sqrt{2}}, -\frac{10}{\sqrt{2}} - 5\right),\quad z_5^\circ = (15, 0),\quad z_6^\circ = \left(\frac{10}{\sqrt{2}} + 5, \frac{10}{\sqrt{2}}\right)$$

$$z_7^\circ = (5,\ 10),\quad z_8^\circ = \left(-\frac{10}{\sqrt{2}},\ \frac{10}{\sqrt{2}}+5\right)$$

The set $K = R^2$. We put

$$\xi(z,t) = z_1^2 + z_2^2 - (15 - {}^1/_2\, t)^2,\ t \geqslant 10,\ T_1 = 10$$

The controls $u_j(t),\ j = 1,\ldots,8,\ 0 \leqslant t \leqslant 10$ , which for $t = T_1 = 10$ ensure that item a) of Condition $B$ is satisfied have the form

$$u_1 = u_2 = ({}^1/_2,\ 0),\ u_3 = u_4 = (0,\ {}^1/_2),\ u_5 = u_6 = u_7 = (-{}^1/_2,\ 0),\ u_8 = (0,\ -{}^1/_2)$$

while the controls $u_j(t),\ j = 1,\ldots,8$ , which ensure that items b) and c) of Condition $B$ are satisfied have the form

$$u_1(t) = ({}^1/_2,\ 0),\ u_2(t) = (\gamma,\ \gamma),\ u_3(t) = (0,\ {}^1/_2)$$
$$u_4(t) = (-\gamma,\ \gamma),\ u_5(t) = (-{}^1/_2,\ 0),\ u_6(t) = (-\gamma,\ -\gamma)$$
$$u_7(t) = (0,\ -{}^1/_2),\ u_8(t) = (\gamma,\ -\gamma),\ 10 \leqslant t \leqslant 22,\ T_2 = 22\ (\gamma = 1/2\sqrt{2})$$

*Example 2.* In Eq.(1) let

$$n = 4,\quad C_i = \begin{Vmatrix} 0 & E \\ 0 & 0 \end{Vmatrix},\quad u_i = \begin{Vmatrix} 0 \\ \mathbf{u}_i \end{Vmatrix},\quad v = \begin{Vmatrix} 0 \\ \mathbf{v} \end{Vmatrix},\quad \mathbf{u}_i \in S_{\rho_i}^2(0)$$
$$\mathbf{v} \in S_\sigma^2(0),\quad M_i^1 = \left\{\begin{Vmatrix} 0 \\ \xi \end{Vmatrix},\ \xi \in R^2\right\},\quad L_i^1 = \left\{\begin{Vmatrix} \eta \\ 0 \end{Vmatrix},\ \eta \in R^2\right\}$$
$$M_i^2 = \left\{\begin{Vmatrix} \eta \\ 0 \end{Vmatrix},\ \eta \in S_{l_i}^2(0)\right\};\quad l_i = 0,\quad \rho_i = \sigma,\quad i = 1,\ldots,k$$
$$\rho_j < \sigma,\ l_j > 0,\ j = k+1,\ldots,m$$

($E$ is a unit 2 X 2 matrix). We put $z_i = (z_{i1},\ z_{i2}),\ z_{i1},\ z_{i2} \in R^2,\ i = 1,\ldots,m.$ Assumption 1 is satisfied in the case in hand; according to (2)

$$\lambda(i,t,\tau,v,z_i^\circ,0) = [(z_{i1}^\circ + tz_{i2}^\circ,\ (t-\tau)\,\mathbf{v}) + ((z_{i1}^\circ + z_{i2}^\circ t,\ (t-\tau)\,\mathbf{v})^2 +$$
$$\|z_{i1}^\circ + z_{i2}^\circ t\|^2 (\sigma^2 (t-\tau)^2 - (\mathbf{v},\ \mathbf{v})\,(t-\tau)^2))^{1/2}] / \|z_{i1}^\circ + z_{i2}^\circ t\|^2$$
$$i = 1,\ldots,k,\ V(t,T,z^\circ) \in K(t,T,z^\circ) =$$
$$\left\{z : \left(z,\ \frac{z_{i1}^\circ + z_{i2}^\circ T}{\|z_{i1}^\circ + z_{i2}^\circ T\|}\right) > (t-T)\,t\sigma - \frac{\|z_{i1}^\circ + z_{i2}^\circ T\|}{2},\quad i = 1,\ldots,k\right\}$$

We present a typical case of set $N(t)$ working its way through set $V(t,T,z^\circ)$ by set $N(t)$ (inside $K(t,T,z^\circ)$). We put $k = 2,\ m = 4;\ \rho_i = \sigma = 1,\ i = 1,2,\ \rho_j = {}^1/_2,\ l_j = 2,\ j = 3,4;\ z_{11}^\circ = (0,\ -3),\ z_{12}^\circ = (0,\ {}^1/_2),\ z_{21}^\circ = (0,2),$ $z_{22}^\circ = (-{}^1/_8,\ -{}^4/_8),\ z_{31}^\circ = (-4\,{}^1/_2,\ -1),\ z_{32}^\circ = (4,\ 0),\ z_{41}^\circ = (-4\,{}^1/_2,\ 3),\ z_{42}^\circ = (4,\ -{}^3/_4)$. Inspecting $T > 0$ and investigating the possibility of solving the capture problem for domain $K(t,T,z^\circ)$, we note that it is solvable, for example, when $T = 3$, for

$$K(t,\ 3,\ z^\circ) = \{z : z \in R^2,\ (z,\ \omega_1) > (t-3)t - {}^3/_4$$
$$(z,\ \omega_2) > (t-3)\,t - \frac{\sqrt{17}}{4}\Big\},\quad \xi(z,t) = z_1 - 4\,{}^1/_2 + 4t + \frac{t^2}{4}$$
$$t \geqslant 0,\ T_1 = 0,\ \omega_1 = (0,\ -1),\ \omega_2 = (\chi,\ 4\chi),\ \chi = -1/17\sqrt{17}$$

The controls $\mathbf{u}_3(t),\ \mathbf{u}_4(t),\ t \in [0,\ 3]$, ensuring that Condition $B$ is satisfied have the form $\mathbf{u}_3(t) = \mathbf{u}_4(t) = ({}^1/_2,\ 0)$.

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# ON A GAME OF OPTIMAL PURSUIT OF AN OBJECT BY TWO OTHERS*

A.G. PASHKOV and S.D. TEREKHOV

A game problem of the simple pursuit of one object by two others, the former having a speed advantage , is analyzed. The duration of the game is fixed. The payoff functional is the distance between the object being pursued and the pursuer closest to it when the game terminates. Similar problems were examined in /1—7/.

The motion of pursuers $P_i\,(y^{(i)})$ is described by the equations

$$y_1^{\cdot(i)} = u_1^{(i)},\ y_2^{\cdot(i)} = u_2^{(i)},\ |u^{(i)}| \leqslant \mu,\ \mu > 0 \tag{1}$$
$$(y^{(i)} = \{y_1^{(i)}, y_2^{(i)}\},\ u^{(i)} = \{u_1^{(i)}, u_2^{(i)}\},\quad i = 1, 2)$$

The object $E\,(z)$ being pursued moves in accordance with the equations

$$z_1^{\cdot} = v_1,\ z_2^{\cdot} = v_2;\ |v| \leqslant \nu,\ \nu > \mu\ (v = \{v_1, v_2\}). \tag{2}$$

Here $u^{(i)}$, $v$ are the control vector. The time the game ends $t = \vartheta$ is fixed. The game's payoff $\gamma$ is the distance between the object being pursued and the pursuer closest to it at the instant $t = \vartheta$, i.e.,

$$\gamma = \min\ (z_1(\vartheta) - y_1^{(i)}(\vartheta))^2 + (z_2(\vartheta) - y_2^{(i)}(v))^2]^{1/2} \tag{3}$$

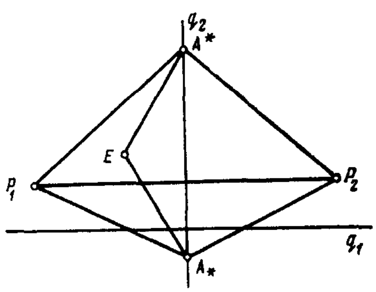


Fig.1

Henceforth we will assume that $|P_1^\circ P_2^\circ| > 0$. The case $P_1^\circ = P_2^\circ$ will be considered separately. In a plane we set up a fixed rectangular system of coordinates with axes $q_1$ and $q_2$. We direct the abscissa axis $q_1$ from the initial position of the first pursuer $P_1^\circ\,(y_0^{(1)})$ to the initial position of the second pursuer $P_2^\circ\,(y_0^{(2)})$. We direct the ordinate axis $q_2$ through the midpoint of the segment $[P_1^\circ P_2^\circ]$ perpendicular to it, so as to obtain a right-oriented system of coordinates (Fig.1). The domain of attainability $G^{(i)}\,(t, y^{(i)}, \vartheta)$ of the objects $P_i$ $(i = 1, 2)$ from the position $\{t, y^{(i)}(t)\}$ by the instant $t = \vartheta$ is a circle of radius $r\,(t) = \mu\,(\vartheta - t)$ with centre at the point $\{y^{(i)}(t)\}$. The domain of attainability $G\,(t, z, \vartheta)$ of the object $E$ from position $\{t, z\,(t)\}$ is a circle of radius $R\,(t) = \nu\,(\vartheta - t)$ with centre at the point $\{z\,(t)\}$. Suppose that at some instant $t$ the object $P_i\ (i = 1, 2)$ is at the position $\{y_1^{(i)}(t), y_2^{(i)}(t)\}$, $y_1^{(1)}(t) = -y_1^{(2)}(t)$, $y_2^{(1)}(t) = y_2^{(2)}(t)$. At the instant $t$ the object $E$ is at position $\{z_1(t), z_2(t)\}$. The attainability domain $G\,(t, z\,(t), \vartheta)$ of the one being pursued intersects the axis $q_2$ at the points $A^*\,(0, q^*)$ and $A_*\,(0, q_*)$ (Fig.1)

$$q^* = z_2\,(t) + ((\nu\,(\vartheta - t))^2 - z_1^2\,(t))^{1/2} \tag{4}$$
$$q_* = z_2\,(t) - ((\nu\,(\vartheta - t))^2 - z_1^2\,(t))^{1/2}$$

We see that the distances between the pursuers $P_i$ and the points $A^*, A_*$ satisfy the following relations: $\operatorname{sign}\,(|P_iA^*| - |P_iA_*|) = \operatorname{sign}\,(z_2\,(t) - y_2^{(i)}\,(t))$.

It can be shown that the optimal programmed strategy for object $E$ to evade pursuers $P_i$ at the instant $t = \vartheta$ from a specified initial position $\{t_0, z_0\}$ will be the extremal control $v\,(t)$ $(t_0 \leqslant t \leqslant \vartheta)$ directed towards the point $A^*$ if

$$z_2\,(t_0) - y_2^{(i)}\,(t_0) > 0$$

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